Poincaré and Prawitz on Mathematical Induction

Yacin Hamami[1]

Abstract: Poincaré and Prawitz have both developed an account of how one can acquire knowledge through reasoning by mathematical induction. Surprisingly, their two accounts are very close to each other: both consider that what underlies reasoning by mathematical induction is a certain chain of inferences by modus ponens 'moving along', so to speak, the well-ordered structure of the natural numbers. Yet, Poincaré's central point is that such a chain of inferences is not sufficient to account for the knowledge acquisition of the universal propositions that constitute the conclusions of inferences by mathematical induction, as this process would require to draw an infinite number of inferences. In this paper, we propose to examine Poincaré's point – that we will call the *closure issue* – in the context of Prawitz's framework where inferences are represented as operations on grounds. We shall see that the closure issue is a challenge that also faces Prawitz's own account of mathematical induction and which points out to an epistemic gap that the chain of modus ponens cannot bridge. One way to address the challenge is to introduce suitable additional inferential operations that would allow to fill the gap. We will end the paper by sketching such a possible solution.

Keywords: Poincaré, Prawitz, mathematical induction

1 Introduction

Poincaré and Prawitz share a common philosophical interest in the nature of reasoning by mathematical induction. For Poincaré, mathematical induction constitutes mathematical reasoning *par excellence* and thereby takes central stage in his philosophical analysis of mathematical reasoning (Poincaré, 1894) as well as in his critical discussion of the role of logic in the foundations of mathematics (Poincaré, 1905, 1906). For Prawitz, mathematical induction is the archetypal example of a deductive reasoning principle that

[1]The author would like to thank the participants at Logica 2014 as well as Jean Paul van Bendegem, John Mumma, and Göran Sundholm for stimulating discussions relative to the content of this paper. The author is a doctoral fellow of the Research Foundation Flanders (FWO).

governs inferences that are valid but not *logically* valid – i.e., whose conclusions are not logical consequences of their premisses in the Bolzano-Tarski sense – and thereby appears as the main example of application of his recent account of the validity of deductive inferences (Prawitz, 2009, 2012, 2013).

In their respective analyses, Poincaré and Prawitz are both concerned with the *epistemological* dimension of reasoning by mathematical induction. Poincaré (1894) aims to understand the capacity of mathematical induction to *extend* mathematical knowledge,[2] a feature that would distinguish it from syllogistic reasoning which "can teach us nothing essentially new" (Poincaré, 1894, p. 31).[3] Prawitz (2009, 2012, 2013) aims to provide an account of the validity of deductive inferences that would explain how one can acquire knowledge by drawing valid inferences. When applied to mathematical induction, such an account would explain how one can acquire knowledge by drawing mathematical induction inferences.

Interestingly, the two accounts developed by Poincaré and Prawitz in order to explain how reasoning by mathematical induction can generate (new) knowledge are very close to each other. More specifically, both consider that what underlies an inference by mathematical induction – i.e., an inference of the following form.

$$\frac{\begin{array}{l} H(0) \\ H(p) \to H(p+1) \end{array}}{H(n)}$$

where n and p are arbitrary natural numbers – is a chain of n inferences by modus ponens which can be described in metaphorical terms as 'starting' from the number 0 and 'moving along' the well-ordered structure of the natural numbers up to the number n: the first inference consists in deducing $H(1)$ from the premisses $H(0)$ and $H(0) \to H(1)$, the p^{th} inference $(p > 0)$ consists in deducing $H(p)$ from $H(p-1)$ and $H(p-1) \to H(p)$, and the n^{th} inference consists in deducing $H(n)$ from $H(n-1)$ and $H(n-1) \to H(n)$.[4] Thus, such an account of reasoning by mathematical induction explains how one can deduce the proposition $H(n)$ for any $n \in \mathbb{N}$ from the two premisses $H(0)$ and $H(p) \to H(p+1)$.

[2] Poincaré speaks of the "creative virtue" of mathematical reasoning (Poincaré, 1894, p. 32).

[3] See (Detlefsen, 1992) and (Heinzmann, 1995) for interpretations that take as central the idea that the main difference between mathematical reasoning and logical reasoning for Poincaré is an epistemological one.

[4] In this paper, we will refer to this inferential process as the *chaining operation*.

Yet, at this stage, Poincaré's and Prawitz's analyses diverge in an important way. On the one hand, Prawitz considers that, insofar as the above chaining operation allows to reach any $n \in \mathbb{N}$, it allows to establish that $H(n)$ holds for an arbitrary $n \in \mathbb{N}$, and so to conclude – by universal generalization – that the universal proposition $\forall n H(n)$ holds, resulting thereby in the knowledge acquisition of the universal proposition $\forall n H(n)$. On the other hand, Poincaré considers that, even though the above chaining operation allows to establish $H(n)$ for any $n \in \mathbb{N}$, it does not allow to establish the universal proposition $\forall n H(n)$, as this would require to draw an *infinite* number of inferences. Poincaré concludes that, to obtain knowledge of the universal proposition $\forall n H(n)$, an additional element is required and this is precisely where Poincaré appeals to his own notion of *intuition*.

The aim of this paper is to understand what lies behind this divergence. To this end, we will examine what we will call Poincaré's *closure issue* – i.e., the incapacity of the chaining operation to yield knowledge of the universal proposition $\forall n H(n)$ – within Prawitz's framework. We shall see that Prawitz's account of the validity of inferences offers a very suitable framework for formulating precisely the closure issue. In turn, this will allow us to examine the impact of Poincaré's closure issue on Prawitz's own account of mathematical induction. We will see that the closure issue points out to a certain gap in the inferential operations necessary for the agent to acquire knowledge of the universal proposition $\forall n H(n)$. Thus, our analysis will show a mutually fruitful interaction between Poincaré's and Prawitz's analyses of mathematical induction: on the one hand, Prawitz's account of the validity of inferences provides a framework to formulate and analyze in a precise way the main point of Poincaré's analysis; on the other hand, Poincaré's analysis yields a central insight that can benefit directly to Prawitz's own account of mathematical induction.

The paper is organized as follows. We will begin by presenting Poincaré's and Prawitz's accounts of mathematical induction respectively in sections 2 and 3. In section 4, we will present Poincaré's closure issue, we will show how it can be formulated within Prawitz's framework in which inferences are represented as operations on grounds and we will analyze its impact on Prawitz's account of mathematical induction. We shall see that the closure issue points out to an epistemic gap that the chain of inferences by modus ponens alone cannot bridge. One way to overcome this is to introduce additional inferential operations that would allow to fill the gap. In section 5, we will sketch such a possible solution by introducing two additional inferential operations that would allow to fill the gap.

2 Poincaré on mathematical induction

Poincaré is well known for his opposition against philosophical views that attribute a prominent role for logic in mathematics, views that emerge from the early development of modern logic and set theory in the context of the foundations of mathematics at the beginning of the 20th century.[5] What is maybe less known is that Poincaré already wrote a philosophical study on the nature of mathematical reasoning as early as 1894,[6] i.e., before most of the developments in logic and set theory that he will discuss in (Poincaré, 1905, 1906), and *a fortiori* before he came aware of them. A central theme of (Poincaré, 1894) was already to compare and contrast mathematical reasoning with logical reasoning – where logical reasoning was equated by Poincaré with syllogistic reasoning. Poincaré took as the starting point of his study that logical reasoning is epistemically sterile while mathematical reasoning has a certain "creative virtue" or, in other words, that mathematical reasoning can *extend* mathematical knowledge while logical reasoning cannot. His aim was to provide an explanation of this epistemological difference between the two types of reasoning. To this end, Poincaré suggested to analyze mathematical reasoning in the branch of mathematics where: "mathematical thought [...] has remained pure, that is, in arithmetics" (Poincaré, 1894, p. 34). Poincaré continued by examining some of the most elementary proofs of arithmetic, namely the proofs of the basic properties of addition (associativity, commutativity) and multiplication (distributivity, commutativity). Poincaré noticed a uniform principle of reasoning present in all these proofs, and this principle is precisely mathematical induction. Poincaré concluded that:

> If we look closely, at every step we meet again this mode of reasoning, either in the simple form we have just given it, or under a form more or less modified.

[5] See (Goldfarb, 1988) for a discussion of the arguments that Poincaré developed in his debate 'against the logicians', among whom figure Cantor, Peano, Russell, Zermelo, and Hilbert. For an overview of Poincaré's work in the philosophy of mathematics and the philosophy of science, we refer the reader to (Heinzmann & Stump, 2014).

[6] We refer here to the paper (Poincaré, 1894) entitled 'Sur la Nature du Raisonnement Mathématique' ('On the Nature of Mathematical Reasoning'). The paper has been reprinted as the first chapter of the book (Poincaré, 1902) entitled *Science et Hypothèse* (*Science and Hypothesis*), which has been translated into English in (Poincaré, 1929). All the quotations from (Poincaré, 1894) in the present paper are taken from the translation provided in (Poincaré, 1929).

> Here then we have the mathematical reasoning *par excellence*, and we must examine it more closely. (Poincaré, 1894, p. 37)

The conception of reasoning by mathematical induction that Poincaré pushes forwards is then spelled out in the following passage:

> The essential characteristic of reasoning by recurrence is that it contains, condensed, so to speak, in a single formula, an infinity of syllogisms.
>
> That this may the better be seen, I will state one after another these syllogisms which are, if you will allow me the expression, arranged in 'cascade.'
>
> These are of course hypothetical syllogisms.
>
> The theorem is true of the number 1.
>
> Now, if it is true of 1, it is true of 2.
>
> Therefore it is true of 2.
>
> Now, if it is true of 2, it is true of 3.
>
> Therefore it is true of 3, and so on. (Poincaré, 1894, p. 37)

Three central points of Poincaré's conception of mathematical induction are presented here. Firstly, Poincaré considers that reasoning by mathematical induction is composed of 'smaller' inferential steps that take the form of *hypothetical syllogisms*. What Poincaré means by hypothetical syllogisms are simply inferences by modus ponens which, in the context of mathematical induction, take the following form:

$$\frac{\begin{array}{l} H(p) \\ H(p) \to H(p+1) \end{array}}{H(p+1)}$$

where p denotes an arbitrary natural number. Secondly, these hypothetical syllogisms are organized in a 'cascade'. If we call the above inference or hypothetical syllogism $\mathbf{I}_p$, this means that an inference by mathematical induction consists in *chaining* hypothetical syllogisms of the above form in the sequence $\mathbf{I}_1, \mathbf{I}_2, \ldots, \mathbf{I}_p, \ldots$ etc. Thirdly, and this is maybe the most important point, insofar as such a sequence of hypothetical syllogisms goes *ad infinitum*, an inference by mathematical induction requires to draw an *infinity* of hypothetical syllogisms. As we shall see in section 4, it is precisely this appeal to infinity that lies at the heart of the epistemological difference between mathematical reasoning and logical reasoning in Poincaré's view.

3 Prawitz on mathematical induction

In his most recent work (Prawitz, 2009, 2012, 2013), Prawitz has developed a new account of the validity of deductive inferences that aims to explain how one can acquire knowledge by drawing valid inferences. To this end, Prawitz has proposed to re-conceptualize the notions of inference and the validity thereof in a way that such a desideratum comes out as a conceptual truth. More specifically, Prawitz proposes to think of an *inference* as comprising not only a set of premisses and a conclusion, but also an *operation*[7] that acts on *grounds* for the premisses and which hopefully would yield a ground for the conclusion. An inference is then said to be *valid* in case such an operation does indeed yield a ground for the conclusion when applied to grounds for the premisses. Given such a re-conceptualization, the above desideratum becomes immediately fulfilled: if we consider that one has knowledge of a given proposition when one has a ground for it,[8] then by drawing a valid inference for which one has grounds for its premisses, one automatically obtains a ground for its conclusion and thereby acquires knowledge of it. Of course, to complete such an account requires to make precise the notions of operation and ground, and we shall now see how this is spelled out.

In Prawitz's framework, the notion of ground is intimately connected to the one of *meaning*:

> The line that I shall take is [...] roughly that the meaning of a sentence is determined by what counts as a ground for the judgement expressed by the sentence. Or expressed less linguistically: it is constitutive for a proposition what can serve as a ground for judging the proposition to be true. From this point of view I shall specify for each compound form of proposition expressible in first order languages what constitutes a ground for an affirmation of a proposition of that form. (Prawitz, 2009, pp. 191–192)

Prawitz (2009) provides such specifications for the propositions of first-order logic that are formed using conjunction, implication and universal quantification. The specifications for the different compound forms follow the same scheme: one should specify how a ground for the compound

[7] We sometimes use equivalently the term *inferential operation*.

[8] This is the view of knowledge that Prawitz focuses on in this context (see Prawitz, 2012, p. 890).

proposition is obtained from its components, and this is achieved by specifying the *grounding operation* associated to each compound form. For instance, in the case of conjunction, Prawitz proposes the following specification: α is a ground for the conjunction $\varphi \wedge \psi$ if and only if $\alpha = \wedge G(\beta, \gamma)$ for some β and γ such that β is a ground for φ and γ is a ground for ψ. In this specification, the *conjunction grounding operation* $\wedge G$, specifies how grounds for propositions of the form $\varphi \wedge \psi$ are formed given grounds for the components φ and ψ.

In order to be able to deal with inferences drawn from assumptions, Prawitz introduces a distinction between *saturated* and *unsaturated* grounds. An unsaturated ground $\alpha(\xi_1, \ldots, \xi_n)$ for a proposition φ under the assumptions $\varphi_1, \ldots, \varphi_n$ is a function that takes as arguments saturated grounds $\beta_1, \ldots, \beta_n$ for $\varphi_1, \ldots, \varphi_n$ and yields a saturated ground $\alpha(\beta_1, \ldots, \beta_n)$ for φ. Furthermore, in order to deal with inferences involving open propositions, an additional distinction is introduced between *closed* and *open* grounds. An open ground $\alpha(x)$ is simply a ground for an open proposition $\varphi(x)$, where the variable(s) in parentheses denotes the free variable(s) in the considered open proposition. Similarly, a closed ground is a ground for a closed proposition. It is important to notice that the two distinctions saturated-unsaturated and closed-open are independent from each other – one can have closed and open saturated grounds as well as closed and open unsaturated grounds.

The distinction between saturated and unsaturated grounds enables to formulate the *implication grounding operation* $\rightarrow G$ which specifies how a saturated ground for a proposition $\varphi \rightarrow \psi$ is formed from an unsaturated ground $\beta(\xi)$ for φ under the assumption ψ: α is a ground for $\varphi \rightarrow \psi$ if and only if $\alpha = \rightarrow G_\xi(\beta(\xi))$ where $\beta(\xi)$ is an unsaturated ground for ψ under the assumption φ. Furthermore, the distinction between closed and open grounds enables to formulate the *universal grounding operation* $\forall G$ which specifies how a closed ground for a proposition of the form $\forall x \varphi(x)$ is formed from an open ground $\beta(x)$ for the open proposition $\varphi(x)$: α is a ground for $\forall x \varphi(x)$ if and only if $\alpha = \forall G_x(\beta(x))$ where $\beta(x)$ is an open ground for $\varphi(x)$.

The other key notion of Prawitz's framework is the one of *operation* on grounds. The three grounding operations $\wedge G$, $\rightarrow G$ and $\forall G$ that we have just defined are typical examples of operations on grounds. Indeed, those operations can be seen as corresponding closely to certain inference rules, namely the introduction rules for conjunction, implication and universal quantification in Gentzen's system of natural deduction. Thus, we can now see how inferences that are made according to those inference rules are represented and categorized as valid in Prawitz's account. For instance, drawing an inference with premisses φ and ψ and conclusion $\varphi \wedge \psi$ according to the introduction rule for conjunction is represented as performing the operation $\wedge G$ on grounds β and γ respectively for φ and ψ which results in the obtention of a ground $\gamma = \wedge G(\beta, \gamma)$. This inference is categorized as valid insofar as $\gamma = \wedge G(\beta, \gamma)$ is indeed a ground for $\varphi \wedge \psi$ in virtue of the meaning of $\varphi \wedge \psi$, i.e., in virtue of what constitutes a ground for $\varphi \wedge \psi$.

The interest of Prawitz's account lies in its capacity to define more complex inferential operations, and thereby to account for more complex inferences, than these primitive ones. In particular, we now have all the elements to spell out how inferences by *modus ponens* and by *mathematical induction* – the two types of inferences central to the subject matter of this paper – can be represented within this framework. To this end, we should specify the inferential operations associated to these two types of inferences.

Modus ponens is an inference schema with premisses φ and $\varphi \rightarrow \psi$, and conclusion ψ. Prawitz (2009, p. 197) defines the inferential operation MP associated to modus ponens by the following equation

$$\mathsf{MP}(\alpha, \beta(\xi)) = \beta(\alpha)$$

where α is a ground for φ and $\beta(\xi)$ is an unsaturated ground for ψ under the assumption φ.[9] We can immediately see that such an inferential operation, when applied to grounds for φ and $\varphi \rightarrow \psi$, yields a ground for ψ since, by definition of unsaturated grounds, the result of saturating the unsaturated ground $\beta(\xi)$ by a saturated ground α for φ is a saturated ground $\beta(\alpha)$ for ψ. Modus ponens is thus an inference schema that generates valid inferences since the inferential operation MP, when applied to grounds for premisses of the forms φ and $\varphi \rightarrow \psi$, yields a ground for the conclusion ψ.

The inferential operation associated to mathematical induction has been informally described by Prawitz as follows:

> Let us consider the inference form of mathematical induction, in which it is concluded that a sentence $A(n)$ holds for an arbitrary natural number n, having established the induction base

[9] Insofar as the second premiss of an inference by modus ponens is a proposition of the form $\varphi \rightarrow \psi$, the correct way to write a ground for such a proposition is $\rightarrow G_\xi(\beta(\xi))$ where $\beta(\xi)$ is an unsaturated ground for ψ under the assumption φ. For readability reasons, we abuse notation and just write $\beta(\xi)$ instead of $\rightarrow G_\xi(\beta(\xi))$.

> that $A(0)$ holds and the induction step that A holds for the successor n' of any natural number n given that A holds for n. The ground for the induction step may be thought of as a chain of operations that results in a ground for $A(n')$ when applied to a ground for $A(n)$. The operation that is involved in this inference form may roughly be described as the operation which, for any given n, takes the given ground for $A(0)$ and then successively applies the chain of operations given as ground for the induction step n times. (Prawitz, 2013, p. 199)

Prawitz (2012, p. 897) provides a precise formulation of this informal description. In order to state it, it will be useful to introduce some convenient notations. We assume that we are working with the language of first-order Peano Arithmetic. We denote variables of the language by x, y, and constants or numerals by n, p.[10] Prawitz (2012) states mathematical induction as an inference schema with premisses $H(0)$ and $H(y) \to H(y+1)$ and conclusion $H(n)$. We introduce the following notations for grounds of the relevant proposition forms, where '$\alpha : \varphi$' should be read as 'α is a ground for φ':[11]

$$\begin{array}{lcl|lcl} \alpha_0 & : & H(0) & \beta_0(\xi_0) & : & H(1) \text{ under the assumption } H(0) \\ \alpha_n & : & H(n) & \beta_p(\xi_p) & : & H(p+1) \text{ under the assumption } H(p) \\ \alpha(x) & : & H(x) & \beta(y,\xi_y) & : & H(y+1) \text{ under the assumption } H(y) \end{array}$$

We can now state precisely the inferential operation of mathematical induction following (Prawitz, 2012, p. 897):

$$\mathsf{IND}^n_{\xi y}(\alpha_0, \beta(y,\xi_y)) = \begin{cases} \alpha_0 & \text{if } n = 0 \\ \mathsf{MP}\left(\mathsf{IND}^{n-1}_{\xi y}(\alpha_0, \beta(y,\xi_y)), \beta_{n-1}(\xi_{n-1})\right) & \text{if } n > 0 \end{cases}$$

This definition by recursion aims to capture the idea that the inferential operation of mathematical induction consists in a chain of applications of modus ponens as informally described in the previous quote as well as in the introduction of this paper, and corresponding to Poincaré's notion of 'cascade'. Prawitz (2012) shows that the operation $\mathsf{IND}^n_{\xi y}$ does indeed yield a ground

[10] We assume that we have numerals in the language available to denote all the natural numbers. We will use the same notation for the numeral and the number it refers to.

[11] For Prawitz, "Grounds are naturally typed by the propositions they are grounds for" (Prawitz, 2009, p. 193). This aspect is rendered here by introducing specific notations for the type of grounds associated to the different proposition forms that will be involved in the present discussion.

for $H(n)$ given grounds α_0 for $H(0)$ and $\beta(y, \xi_y)$ for $H(y+1)$ under the assumption $H(y)$, and this for any $n \in \mathbb{N}$. As we shall now see, Poincaré agrees on this point, but contests that we can infer from this that the agent can obtain a ground for the universal proposition $\forall x H(x)$ in this way.

4 The closure issue

Both Poincaré and Prawitz consider that the chaining operation underlies inferences by mathematical induction. Furthermore, Poincaré agrees with Prawitz that the chaining operation allows one to acquire knowledge of the proposition $H(n)$, and this for any $n \in \mathbb{N}$, as witnessed by the following passage:

> If instead of showing that our theorem is true of all numbers, we only wish to show it true of the number 6, for example, it will suffice for us to establish the first 5 syllogisms of our cascade; 9 would be necessary if we wished to prove the theorem for the number 10; more would be needed for a larger number; but, however great this number might be, we should always end by reaching it, and the analytic verification would be possible. (Poincaré, 1894, p. 38)

This observation of Poincaré – that the chaining operation allows one to reach knowledge of the proposition $H(n)$ however great n might be – can be shown formally within Prawitz's framework:

Proposition 1 *If an agent has at her disposal (i) a ground α_0 for $H(0)$ and a ground $\beta(y, \xi_y)$ for $H(y+1)$ under the assumption $H(y)$ and (ii) the inferential operation* MP*, then the agent can obtain a ground for $H(n)$ by drawing n inferences, and this for any $n \in \mathbb{N}$.*

Proof. This can be shown by a simple induction on n.
Base case: The agent can obtain a ground for $H(1)$ by applying MP to the grounds α_0 for $H(0)$ and $\beta_0(\xi_0)$ for $H(1)$ under the assumption $H(0)$, as this would yield the ground $\beta_0(\alpha_0)$ which is a ground for $H(1)$.
Induction case: Assume that the agent can obtain a ground for $H(n)$ by drawing n inferences. Let α_n be such a ground for $H(n)$. The agent possesses then a ground α_n for $H(n)$ and a ground $\beta_n(\xi_n)$ for $H(n+1)$ under the assumption $H(n)$ (obtained from $\beta(y, \xi_y)$ by universal instantiation). Thus, the agent can apply MP to α_n and $\beta_n(\xi_n)$, obtaining thereby a ground $\beta_n(\alpha_n)$ for $H(n+1)$. □

However, Poincaré considers that the chaining operation is insufficient to reach knowledge of the universal proposition $\forall x H(x)$, as Poincaré puts it:

> And yet, however far we thus might go, we could never rise to the general theorem, applicable to all numbers, which alone can be the object of science. To reach this, an infinity of syllogisms would be necessary; it would be necessary to overleap an abyss that the patience of the analyst, restricted to the resources of formal logic alone, never could fill up. (Poincaré, 1894, p. 38)

Thus, what prevents the chaining operation to reach knowledge of the universal proposition $\forall x H(x)$, according to Poincaré, is precisely that this would require to draw an *infinite* number of inferences, and this is what we refer to as the *closure issue*. As expressed in the last sentence of the above passage, Poincaré attributes this limitation to the resources of logical reasoning.

The closure issue can be established formally within Prawitz's framework. To this end, we first need to prove the following lemma:

Lemma 1 *If an agent only has at her disposal (i) a ground α_0 for $H(0)$ and a ground $\beta(y, \xi_y)$ for $H(y+1)$ under the assumption $H(y)$ and (ii) the inferential operation* MP*, then the only grounds the agent can obtain by drawing n inferences are grounds for the propositions $H(1), \ldots, H(n)$.*

Proof. The proof goes by induction on the number of inferences n.
Base case: We have to show that the only ground the agent can obtain by drawing a single inference is a ground for $H(1)$. We first notice that, in order to apply the inferential operation MP, the agent needs to be in possession of a ground for a proposition φ and a ground for a proposition of the form $\varphi \to \psi$. By assumption, the two grounds that the agent has initially are grounds α_0 for $H(0)$ and $\beta(y, \xi_y)$ for $H(y+1)$ under the assumption $H(y)$. Thus, the only grounds to which the agent can apply MP are α_0 for $H(0)$ and $\beta_0(\xi_0)$ for $H(1)$ under the assumption $H(0)$, and this would yield $\beta_0(\alpha_0)$ which is a ground for $H(1)$. Hence, the only ground the agent can obtain by drawing a single inference is a ground for $H(1)$.
Induction case: Assume that the only grounds the agent can obtain by drawing n inferences are grounds for the propositions $H(1), \ldots, H(n)$. Again, in order to apply the inferential operation MP, the agent needs to be in possession of a ground for a proposition φ and a ground for a proposition of the form $\varphi \to \psi$. By induction hypothesis, the only grounds the agent can

possess after n inferences are grounds for the propositions $H(1), \ldots, H(n)$, in addition to the grounds α_0 for $H(0)$ and $\beta(y, \xi_y)$ for $H(y+1)$ under the assumption $H(y)$ that the agent possessed initially. Thus, the only pairs of grounds to which the agent can apply the inferential operation MP are $(\alpha_0, \beta_0(\xi_0)), \ldots, (\alpha_n, \beta_n(\xi_n))$. It follows that the ground the agent can obtain by applying MP in this case is necessarily a ground for one of the propositions $H(1), \ldots, H(n+1)$. □

Poincaré's closure issue is a direct consequence of Lemma 1 and can be formalized as follows:

Proposition 2 (**Closure Issue**) *If an agent only has at her disposal (i) a ground α_0 for $H(0)$ and a ground $\beta(y, \xi_y)$ for $H(y+1)$ under the assumption $H(y)$ and (ii) the inferential operation* MP, *then the agent cannot obtain a ground $\alpha(x)$ for $H(x)$ by drawing a finite number of inferences.*[12]

Proof. This follows directly from Lemma 1. □

This proposition is the direct formalization of Poincaré's closure issue within Prawitz's framework. Given that Prawitz's informal description of the inferential operation of mathematical induction in the quote of section 3 is similar to the 'cascade' operation provided by Poincaré, the closure issue does apply as well to Prawitz's account of mathematical induction. Indeed, we can even show that the inferential operations $\mathsf{IND}^n_{\xi y}$ defined in section 3 provide insufficient means to obtain a ground for the universal proposition $\forall x H(x)$:

Proposition 3 *If an agent only has at her disposal (i) a ground α_0 for $H(0)$ and a ground $\beta(y, \xi_y)$ for $H(y+1)$ under the assumption $H(y)$ and (ii) the inferential operations $\mathsf{IND}^n_{\xi y}$ for all $n \in \mathbb{N}$, then the agent cannot obtain a ground $\alpha(x)$ for $H(x)$ by drawing a finite number of inferences.*

Proof. This can be proved by induction in a similar fashion as in Lemma 1 and Proposition 2. □

[12] Since one can easily pass from a ground $\alpha(x)$ for $H(x)$ to a ground $\beta = \forall G_x(\alpha(x))$ for $\forall x H(x)$ and the other way around by applying the inferential operations corresponding respectively to the introduction and elimination rules for universal quantification in natural deduction, we do not necessarily distinguish between the two cases in the present discussion.

5 A possible solution

One could expect that Poincaré's conclusion from the closure issue is a skeptical one – namely that it is simply impossible to acquire knowledge of the universal proposition $\forall x H(x)$. This is not the case. Poincaré does think that it is possible to acquire knowledge of the universal proposition $\forall x H(x)$. His conclusion from the closure issue is rather that *logical* or *syllogistic* reasoning provides *insufficient* means to acquire knowledge of $\forall x H(x)$. Poincaré explains then our capacity to acquire knowledge of $\forall x H(x)$ by appealing to another mean of knowledge acquisition, namely *intuition*:

> Why then does this judgment force itself upon us with an irresistible evidence? It is because it is only the affirmation of the power of the mind which knows itself capable of conceiving the indefinite repetition of the same act when once this act is possible. The mind has a direct intuition of this power, and experience can only give occasion for using it and thereby becoming conscious of it. (Poincaré, 1894, p. 39)

Unfortunately, this passage contains all what Poincaré has to say on the role of intuition in mathematical induction in (Poincaré, 1894). In the remainder of this paper, our goal is not to propose a detailed analysis of what Poincaré could mean here by intuition. Rather, we propose to sketch a solution to the closure issue based on a straightforward interpretation of the above passage.

Poincaré says that the mind has the capacity to conceive "the indefinite repetition of the same act when once this act is possible". Taken at face value, this statement states that the mind is in possession of an inferential operation that allows to take the limit or closure, so to speak, of the chaining operation. This could be formalized in Prawitz's framework by introducing an inferential operation $\mathsf{IND}^{\infty}_{\xi y}$ which would correspond to what would be achieved by carrying out the chaining operation *ad infinitum*:[13]

$$\mathsf{IND}^{\infty}_{\xi y}(\alpha, \beta(y, \xi_y)) = \lim_{n\to\infty} \mathsf{IND}^{n}_{\xi y}(\alpha, \beta(y, \xi_y)) = \bigcup_{n\in\mathbb{N}} \mathsf{IND}^{n}_{\xi y}(\alpha, \beta(y, \xi_y))$$

Yet, this inferential operation would still be insufficient to acquire a ground for the universal proposition $\forall x H(x)$, as shown by the following proposition:

[13] We use here the set-theoretic notion of limit. For readability reasons, we identify the ground $\mathsf{IND}^{n}_{\xi y}(\alpha, \beta(y, \xi_y))$ with the singleton $\{\mathsf{IND}^{n}_{\xi y}(\alpha, \beta(y, \xi_y))\}$.

Proposition 4 *If an agent only has at her disposal (i) a ground α_0 for $H(0)$ and a ground $\beta(y, \xi_y)$ for $H(y+1)$ under the assumption $H(y)$ and (ii) the inferential operation $\mathsf{IND}^{\infty}_{\xi y}$, then the agent cannot obtain a ground $\alpha(x)$ for $H(x)$ by drawing a finite number of inferences.*

Proof. The operation $\mathsf{IND}^{\infty}_{\xi y}$ only produces grounds for propositions of the form $H(n)$, and so will never be able to output a ground for a proposition of the form $H(x)$. This can be shown by a straightforward induction on the number of inferences in a similar fashion as in the proof of Lemma 1. $\square$

Nevertheless, by carrying out the operation $\mathsf{IND}^{\infty}_{\xi y}$ on grounds for $H(0)$ and $H(y) \to H(y+1)$, one would already be in possession of grounds for *all* the propositions of the form $H(n)$ with $n \in \mathbb{N}$. To reach a ground for $\forall x H(x)$, it suffices to introduce a further inferential operation which would take as inputs the grounds $(\alpha_n)_{n\in\mathbb{N}}$ for $(H(n))_{n\in\mathbb{N}}$ and which would output a ground for $H(x)$. Surprisingly, such an operation corresponds exactly to what is known as the ω-Rule in the field of proof theory. This inferential operation would take grounds $(\alpha_n)_{n\in\mathbb{N}}$ respectively for the propositions $(H(n))_{n\in\mathbb{N}}$ and be defined by the following equation:

$$\omega\text{-}\mathsf{R}\left((\alpha_n)_{n\in\mathbb{N}}\right) = \alpha(x)$$

We can now show that, when the agent disposes of the inferential operations $\mathsf{IND}^{\infty}_{\xi y}$ and ω-R, she has the capacity to acquire knowledge of $\forall x H(x)$:

Proposition 5 *If an agent has at her disposal (i) a ground α_0 for $H(0)$ and a ground $\beta(y, \xi_y)$ for $H(y+1)$ under the assumption $H(y)$ and (ii) the inferential operations $\mathsf{IND}^{\infty}_{\xi y}$ and ω-R, then the agent can obtain a ground $\alpha(x)$ for $H(x)$ by drawing a finite number of inferences.*

Proof. By applying the inferential operation $\mathsf{IND}^{\infty}_{\xi y}$ to α_0 and $\beta(y, \xi_y)$, the agent will obtain the set of grounds $\bigcup_{n\in\mathbb{N}} \mathsf{IND}^{n}_{\xi y}(\alpha, \beta(y, \xi_y))$. Given that $\mathsf{IND}^{n}_{\xi y}(\alpha, \beta(y, \xi_y))$ is a ground for $H(n)$, the agent is then in possession of a ground α_n for $H(n)$ and this for all $n \in \mathbb{N}$. The agent can then apply the operation ω-R to $(\alpha_n)_{n\in\mathbb{N}}$, obtaining thereby a ground $\alpha(x)$ for $H(x)$. $\square$

6 Conclusion

Poincaré's and Prawitz's accounts of mathematical induction are facing the same challenge raised by the closure issue. This challenge must be addressed for the two accounts to reach their objective, namely to account for knowledge acquisition through reasoning by mathematical induction. We have provided a possible solution to this challenge by introducing two inferential operations $\mathsf{IND}^{\infty}_{\xi y}$ and ω-R. However, this would qualify as a potential solution only insofar as it is *epistemologically acceptable* for the agent to possess and carry out such inferential operations. To determine whether this is so requires an epistemological analysis that is left to further investigations.

References

Detlefsen, M. (1992). Poincaré against the Logicians. *Synthese*, *90*(3), 349–378.

Goldfarb, W. (1988). Poincaré against the Logicists. In W. Aspray & P. Kitcher (Eds.), *History and Philosophy of Modern Mathematics* (pp. 61–81). Minneapolis: University of Minnesota Press.

Heinzmann, G. (1995). *Zwischen Objektkonstruktion und Strukturanalyse. Zur Philosophie der Mathematik bei Jules Henri Poincaré*. Göttingen: Vandenhoek & Ruprecht.

Heinzmann, G., & Stump, D. (2014). Henri Poincaré. In E. N. Zalta (Ed.), *The Stanford Encyclopedia of Philosophy* (Spring 2014 ed.). `http://plato.stanford.edu/archives/spr2014/entries/poincare/`.

Poincaré, H. (1894). Sur la Nature du Raisonnement Mathématique. *Revue de Métaphysique et de Morale*, *2*, 371–384. Reprinted in (Poincaré, 1902). Cited as it appears in the English translation (Poincaré, 1929).

Poincaré, H. (1902). *La Science et l'Hypothèse*. Paris: Ernest Flammarion.

Poincaré, H. (1905). Les Mathématiques et la Logique. *Revue de Métaphysique et de Morale*, *13*(6), 815–835.

Poincaré, H. (1906). Les Mathématiques et la Logique (Suite et Fin). *Revue de Métaphysique et de Morale*, *14*(6), 17–34.

Poincaré, H. (1929). *The Foundations of Science: Science and Hypothesis, The Value of Science, Science and Method*. New York and Garrison, N.Y.: The Science Press.

Prawitz, D. (2009). Inference and Knowledge. In M. Peliš (Ed.), *Logica Yearbook 2008* (pp. 175–192). London: College Publications.

Prawitz, D. (2012). The Epistemic Significance of Valid Inference. *Synthese*, *187*(3), 887–898.

Prawitz, D. (2013). Validity of Inferences. In M. Frauchiger (Ed.), *Reference, Rationality, and Phenomenology: Themes from Føllesdal* (pp. 179–204). Heusenstamm: Ontos Verlag.

Yacin Hamami
Centre for Logic and Philosophy of Science, Vrije Universiteit Brussel
Belgium
E-mail: yacin.hamami@vub.ac.be